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Optimal Bounds for Conduction in Two-Dimensional, Multiphase, Polycrystalline Media

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A complete geometrical characterization is given of the set of all possible effective conductivity tensors for two-dimensional composites made of an arbitrary number of given anisotropic phases. It is rigorously established that any polycrystalline composite formed from an arbitrary (possibly infinite) number of phases can be replaced by a composite formed from only two of the given phases without altering the effective conductivity tensor.

KEY WORDS: Polycrystalline composites; optimal bounds; effective conductivity; homogenization.

1. INTRODUCTION

This paper is concerned with the determination of bounds on the effective conductivity of a multiphase polycrystalline material with arbitrary phase geometry. Historically, attention was first restricted to composites with isotropic phases: Hashin[1] gives a review of this early work. The subsequent developments for polycrystalline media[2] focused on overall bounds on the individual eigenvalues of the effective conductivity tensor and no attempt was made to correlate these eigenvalues.

In a two-dimensional setting, a characterization of the set of all anisotropic conducting materials resulting from the mixture in arbitrary volume fraction of two anisotropic conducting phases was first proposed by Lurie and Cherkaev.[3] A more complete characterization was obtained by Francfort and Murat[4] based on the ideas of H-convergence and compensated compactness developed by Murat and Tartar.[5-10]

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Here we obtain a complete characterization of the set of all possible effective conductivity tensors for two-dimensional composites made of an arbitrary number of given anisotropic phases. The eigenvalues of the conductivity tensors of the phases are assumed to be bounded above and below. Our main result, as anticipated by Lurie and Cherkaev,[3] is that any polycrystalline composite formed from an arbitrary (possibly infinite) number of phases can be replaced by a composite formed from only two of the given phases without altering the effective conductivity tensor.

Lamination is a straightforward process of generating composites from one or two anisotropic phases. Specifically, each material is sliced orthogonal to one of its principal conductivity axes. It is then layered with the same material sliced in the orthogonal direction or with the other material sliced in either direction.

Let (α_1, α_2) and (β_1, β_2) be the respective eigenvalue pairs of the conductivity tensors of phase 1 and phase 2 and assume, with no loss of generality, that

$$\alpha_1 \leqslant \alpha_2, \qquad \beta_1 \leqslant \beta_2, \qquad \alpha_1 \alpha_2 \leqslant \beta_1 \beta_2 \tag{1.1}$$

A straightforward calculation[4] establishes that the eigenvalues λ_1 and λ_2 of the effective conductivity tensors obtained through the aforementioned lamination processes lie on one of the following curves:

$$\lambda_1 \lambda_2 = \alpha_1 \alpha_2, \qquad \text{with} \quad \alpha_1 \leqslant \lambda_1, \lambda_2 \leqslant \alpha_2 \tag{1.2}$$

$$\lambda_1 \lambda_2 = \beta_1 \beta_2, \qquad \text{with} \quad \beta_1 \leqslant \lambda_1, \lambda_2 \leqslant \beta_2 \tag{1.3}$$

$$\begin{gathered}\lambda_1 \text{ (or } \lambda_2) = [(\beta_1 - \alpha_1) \lambda_1 \lambda_2 + (\beta_2 - \alpha_2) \alpha_1 \beta_1]/(\beta_1 \beta_2 - \alpha_1 \alpha_2) \\ \text{with} \quad \alpha_1 \alpha_2 \leqslant \lambda_1 \lambda_2 \leqslant \beta_1 \beta_2\end{gathered} \tag{1.4}$$

$$\begin{gathered}\lambda_1 \text{ (or } \lambda_2) = [(\beta_2 - \alpha_2) \lambda_1 \lambda_2 + (\beta_1 - \alpha_1) \alpha_2 \beta_2]/(\beta_1 \beta_2 - \alpha_1 \alpha_2) \\ \text{with} \quad \alpha_1 \alpha_2 \leqslant \lambda_1 \lambda_2 \leqslant \beta_1 \beta_2\end{gathered} \tag{1.5}$$

or on one of the curves obtained by interchanging the subscripts 1 and 2 on α in (1.4) and (1.5).

It is rigorously proved by Francfort and Murat[4] that an arbitrary conductivity tensor with eigenvalues λ_1 and λ_2 can be obtained as an effective tensor for the mixture of the two original phases if and only if the eigenvalue pair (λ_1, λ_2) lies inside the outermost region of the (λ_1, λ_2) plane bounded by the curves (1.2)–(1.5). Three cases have to be distinguished:

Case 1: $\alpha_1 \leqslant \beta_1$ and $\alpha_2 \leqslant \beta_2$

Case 2: $\alpha_1 < \beta_1$ and $\alpha_2 > \beta_2$

Case 3: $\alpha_1 > \beta_1$ and $\alpha_2 < \beta_2$

Note that, in view of (1.1), these three cases cover all possibilities. Although cases 2 and 3 could be grouped together, we choose to treat them separately because they represent disjoint possibilities. In case 1 the outermost boundary is the union of the curves (1.2)–(1.4), whereas in cases 2 and 3 the outermost boundary is the union of the curves (1.2), (1.3), and (1.5). Thus, if we define $\mathscr{L}$ in case 1 [i.e., when $(\alpha_1-\beta_1)(\alpha_2-\beta_2)\geqslant 0$] as the region between the branches of the hyperbolas

$$\frac{(\beta_1-\alpha_1)\,\lambda_1\lambda_2+(\beta_2-\alpha_2)\,\alpha_1\beta_1}{\beta_1\beta_2-\alpha_1\alpha_2}\leqslant\lambda_1,\,\lambda_2\leqslant\frac{\lambda_1\lambda_2(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_1-\alpha_1)\,\lambda_1\lambda_2+(\beta_2-\alpha_2)\,\alpha_1\beta_1}$$

$$\alpha_1\alpha_2\leqslant\lambda_1\lambda_2\leqslant\beta_1\beta_2 \tag{1.6}$$

or in cases 2 and 3 [i.e., when $(\alpha_1-\beta_1)(\alpha_2-\beta_2)<0$] as the region between the branches of the hyperbolas

$$\frac{(\beta_2-\alpha_2)\,\lambda_1\lambda_2+(\beta_1-\alpha_1)\,\alpha_2\beta_2}{\beta_1\beta_2-\alpha_1\alpha_2}\geqslant\lambda_1,\,\lambda_2\geqslant\frac{\lambda_1\lambda_2(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_2-\alpha_2)\,\lambda_1\lambda_2+(\beta_1-\alpha_1)\,\alpha_2\beta_2}$$

$$\alpha_1\alpha_2\leqslant\lambda_1\lambda_2\leqslant\beta_1\beta_2 \tag{1.7}$$

as illustrated in Fig. 1, then the set of all possible effective conductivity tensors coincides with the set of all tensors whose eigenvalue pair (λ_1, λ_2) lies in $\mathscr{L}$.

Here this result is generalized to two-dimensional composites containing arbitrarily many given anisotropic phases. The set of all possible eigenvalue pairs (λ_1, λ_2) of the effective tensor is found to have a simple geometrical characterization, which is illustrated in Fig. 2 for a three-phase composite. First the eigenvalue pairs corresponding to the given phases are plotted in the $(\lambda_1\lambda_2, \lambda_2)$ plane, together with the associated eigenvalue pairs obtained by interchanging λ_1 and λ_2 (Fig. 2a). Then the convex hull is taken (Fig. 2b) and mapped to the $(\lambda_1\lambda_2, \lambda_1)$ plane (Fig. 2c) and the convex hull of the resulting region is formed (Fig. 2d). The set thus obtained when mapped back to the (λ_1, λ_2) plane represents the set of all possible eigenvalue pairs of two-dimensional composites formed from the given phases.

In the first section of the paper basic facts concerning the theory of homogenization are recalled and the problem is formulated in the mathematical framework of H-convergence. In the second section the results obtained by Francfort and Murat[4] for two-phase materials are reviewed. We then proceed to prove the results announced above for multiphase materials with arbitrarily many phases (cf. Remark 3.6 and

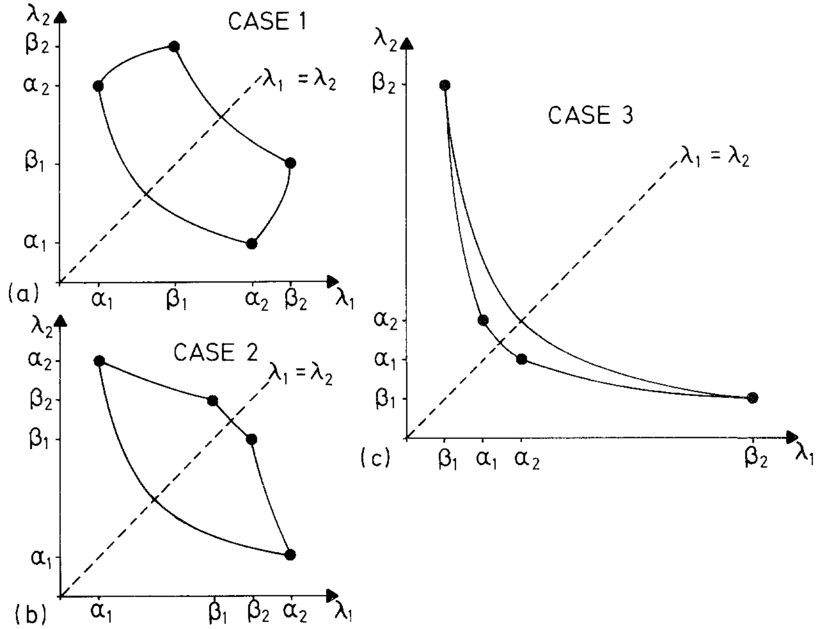

Fig. 1. Typical examples of the region $\mathscr{L}$ in the (λ_1, λ_2) plane for each of the three cases defined in the introduction. The region $\mathscr{L}$ represents the set of all possible eigenvalue pairs (λ_1, λ_2) of effective tensors of composites formed from mixing two phases with conductivity tensor eigenvalue pairs (α_1, α_2) and (β_1, β_2). The curves bounding $\mathscr{L}$ are the outermost curves obtained from (1.2)–(1.5).

Theorem 3.2). The geometrical characterization of the set of all possible effective tensors is also derived (cf. Corollary 3.1).

The crux of the proof is to assume the existence of an effective tensor $\mathbf{A}^0$ that cannot be associated with a composite of any two of the original phases and to infer the existence of two fictitious materials that generate composites whose possible effective conductivity tensors include those of the original phases but exclude $\mathbf{A}^0$. This is absurd, since any composite of the original phases can be regarded as a composite of these two fictitious materials. The idea of introducing fictitious materials to obtain bounds is due to Schulgasser[2] and is embodied in the trajectory method of Bergman.[11]

Throughout the text a few remarks, such as Remarks 3.5 and 3.7, are inserted for the sake of completeness, but are not necessary for an understanding of the rest of the paper. As far as notation is concerned, capital Roman letters denote symmetric second-order tensors or tensor

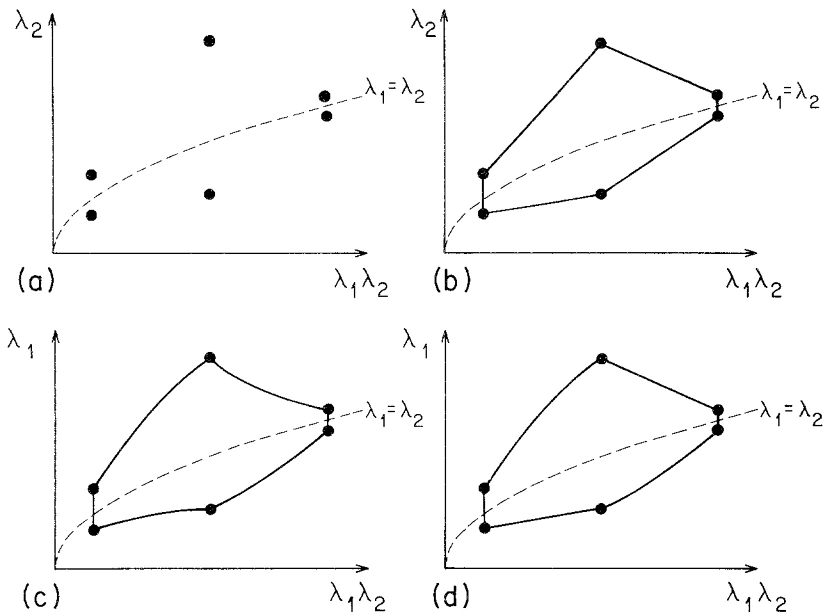

Fig. 2. Geometrical construction of the set of all possible eigenvalue pairs (λ_1, λ_2) of the effective tensor of a three-phase composite. The eigenvalue pairs associated with the conductivity tensors of the given phases are plotted in the $(\lambda_1\lambda_2, \lambda_2)$ plane (Fig. 2a). The convex hull is taken (Fig. 2b) and mapped to the $(\lambda_1\lambda_2, \lambda_1)$ plane (Fig. 2c), and its convex hull is formed (Fig. 2d) to complete the construction. All eigenvalue pairs within the resulting region, and only these eigenvalue pairs, correspond to the eigenvalues of the effective tensor of a composite formed from the three phases.

fields on $\mathbb{R}^2$, capital Greek letters denote sets of such tensors or tensor fields, and capital script letters denote subsets of the upper right open quadrant, $\mathbb{R}_*^+ \times \mathbb{R}_*^+$.

2. A MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider a family of homogeneous and anisotropic conducting materials in our two-dimensional space $\mathbb{R}^2$. They are oriented in a manner such that the principal axes of their respective conductivity tensors coincide. We denote by $(\mathbf{e}_1, \mathbf{e}_2)$ their common principal orthonormal basis. An arbitrary element of the family is characterized by the two eigenvalues, λ_1 and λ_2, of its associated conductivity tensor, which are strictly positive real numbers. Thus the oriented family under consideration can be represented by a subset $\mathscr{S}$ of the upper right open quadrant, $\mathbb{R}_+^* \times \mathbb{R}_+^*$, of the plane: each material is represented by a pair of points, one point whose coordinates are the eigenvalues λ_1 and λ_2 of the conductivity tensor of the material and an associated point whose coordinates are obtained by interchanging λ_1 and λ_2. This pairing of points accounts for the symmetry in Fig. 1 about the line $\lambda_1 = \lambda_2$.

An equivalent definition of the oriented family is the set

$$\Sigma \equiv \{\mathbf{B} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2, (\lambda_1, \lambda_2) \in \mathscr{S}\} \tag{2.1}$$

An arbitrary composite made of materials of the original family has a conductivity tensor field $\mathbf{A}$ of form

$$\mathbf{A}(\mathbf{x}) = {}^t\mathbf{R}(\mathbf{x})\, \mathbf{B}(\mathbf{x})\, \mathbf{R}(\mathbf{x}) \tag{2.2}$$

where for almost every $\mathbf{x}$ of $\mathbb{R}^2$, $\mathbf{B}(\mathbf{x})$ lies in Σ and $\mathbf{R}(\mathbf{x})$, with transpose ${}^t\mathbf{R}(\mathbf{x})$, is an orthogonal matrix, which indicates the orientation of the crystal at the point $\mathbf{x}$.

In order to investigate the macroscopic properties of all such composites we consider a family of measurable conductivity tensor fields $\mathbf{B}^\varepsilon(\mathbf{x})$ with values in Σ and a family of measurable orientation functions $\mathbf{R}^\varepsilon(\mathbf{x})$. The corresponding family of conductivity tensor fields,

$$\mathbf{A}^\varepsilon(\mathbf{x}) = {}^t\mathbf{R}^\varepsilon(\mathbf{x})\, \mathbf{B}^\varepsilon(\mathbf{x})\, \mathbf{R}^\varepsilon(\mathbf{x}) \tag{2.3}$$

has spatial variations on a length scale of ε. The introduction of the orientation functions $\mathbf{R}^\varepsilon(\mathbf{x})$ allows an arbitrary orientation of the local conductivity tensor at each point of the composite.

A possibly inhomogeneous conducting material characterized by $\mathbf{A}^0 \equiv \mathbf{A}^0(\mathbf{x})$ is an effective material for the mixture (and $\mathbf{A}^0$ is the associated effective tensor field) if there exists a subsequence of $\mathbf{A}^\varepsilon$ such that, on any domain Ω of $\mathbb{R}^2$, the solution to an arbitrary conduction problem with $\mathbf{A}^\varepsilon$ as the conductivity tensor field yields a potential field, a current flux, and a local power dissipation that remain close on average, when ε is sufficiently small, to the potential field, the current flux, and the local power dissipation associated with the solution of the same conduction problem with $\mathbf{A}^0$ as the conductivity tensor field.

The mathematical translation of these notions is the notion of H-convergence.

Definition 2.1. If α and β are two strictly positive real numbers, then $M_2(\alpha, \beta)$ is the set of all symmetric second-order tensor fields $\mathbf{A}(\mathbf{x})$ with coefficients in $L_\infty(\mathbb{R}^2)$ such that

$$\alpha \mathbf{I} \leqslant \mathbf{A}(\mathbf{x}) \leqslant \beta \mathbf{I} \tag{2.4}$$

for almost every $\mathbf{x} \in \mathbb{R}^2$.

Definition 2.2. A sequence $\mathbf{A}^\varepsilon$ of elements of $M_2(\alpha, \beta)$ is said to H-converge to a symmetric second-order tensor field $\mathbf{A}^0$ if and only if, on each bounded domain Ω of $\mathbb{R}^2$, all pairs of sequences of vector fields $\mathbf{w}^\varepsilon$ and $\mathbf{q}^\varepsilon$ in $(L^2(\Omega))^2$ for which (i) the identity

$$\mathbf{q}^\varepsilon = \mathbf{A}^\varepsilon \mathbf{w}^\varepsilon \tag{2.5}$$

is satisfied for all ε; (ii) $\mathbf{w}^\varepsilon$ and $\mathbf{q}^\varepsilon$ converge weakly in $(L_2(\Omega))^2$ as ε tends to zero; and (iii) the scalar fields

$$\operatorname{curl} \mathbf{w}^\varepsilon \equiv \partial w_1^\varepsilon/\partial x_2 - \partial w_2^\varepsilon/\partial x_1, \qquad \operatorname{div} \mathbf{q}^\varepsilon \equiv \partial q_1^\varepsilon/\partial x_1 + \partial q_2^\varepsilon/\partial x_2 \tag{2.6}$$

lie in a compact set of $H_{\text{loc}}^{-1}(\Omega)$, have weak limits $\mathbf{w}^0$ and $\mathbf{q}^0$ satisfying

$$\mathbf{q}^0 = \mathbf{A}^0 \mathbf{w}^0 \tag{2.7}$$

Remark 2.1. Physically, the requirement that curl $\mathbf{w}^\varepsilon$ and div $\mathbf{q}^\varepsilon$ lie in compact set of $H_{\text{loc}}^{-1}(\Omega)$ places a constraint on the deviation of the fields $\mathbf{w}^\varepsilon$ from gradients of potentials and of div $\mathbf{q}^\varepsilon$ from fixed source terms. (Sources of current that extend throughout Ω are allowed in this general treatment of the problem.)

This definition is motivated by the following theorem (see Murat[5] or Spagnolo[12] in a similar context.

Theorem 2.1. Let $\mathbf{A}^\varepsilon$ be a family of elements of $M_2(\alpha, \beta)$. There exists a subsequence of $\mathbf{A}^\varepsilon$ that H-converges to an element $\mathbf{A}^0$ of $M_2(\alpha, \beta)$.

Under the hypothesis that $\mathscr{S}$ is a compact subset of the open quadrant, $(\mathbb{R}_+^*)^2$, the particular family of $\mathbf{A}^\varepsilon$'s considered in (2.3) satisfies (2.4) with

$$\alpha = \inf_{(\lambda_1, \lambda_2) \in \mathscr{S}} [\min(\lambda_1, \lambda_2)], \qquad \beta = \sup_{(\lambda_1, \lambda_2) \in \mathscr{S}} [\max(\lambda_1, \lambda_2)] \tag{2.8}$$

Our objective is to characterize the set of all $\mathbf{A}^0$'s that can be obtained as H-limits of sequences of $\mathbf{A}^\varepsilon$'s of the form (2.3). Theorem 2.1 ensures the existence of such $\mathbf{A}^0$'s.

In summary, after possible extraction of subsequences, we are left with the following set of conditions to be satisfied by the sequences of $\mathbf{A}^\varepsilon$'s:

(i) Each $\mathbf{A}^\varepsilon$ has the form

$$\mathbf{A}^\varepsilon = {}^t\mathbf{R}^\varepsilon(\lambda_1^\varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^\varepsilon \mathbf{e}_2 \otimes \mathbf{e}_2)\, \mathbf{R}^\varepsilon \tag{2.9}$$

where:

(ii) $(\lambda_1^\varepsilon, \lambda_2^\varepsilon)$ is a measurable sequence of $\mathscr{S}$-valued pairs satisfying

$$\lambda_1^\varepsilon \leqslant \lambda_2^\varepsilon \tag{2.10}$$

and $\mathscr{S}$ is a compact subset of $(\mathbb{R}_+^*)^2$ which is symmetric under interchange of the coordinates

(iii) $\mathbf{R}^\varepsilon$ is a measurable sequence of orthogonal matrices on $\mathbb{R}^2$

(iv) $\mathbf{A}^\varepsilon$ H-converges

We label these as the (H-$\mathscr{S}$) set of conditions.

Remark 2.2. Since $\mathscr{S}$ is invariant under interchange of the coordinates [i.e., $(\lambda_2, \lambda_1) \in \mathscr{S}$ if and only if $(\lambda_1, \lambda_2) \in \mathscr{S}$], the inequality (2.10) can be satisfied with no loss of generality at the expense of a possible increase of the rotation angle associated with $\mathbf{R}^\varepsilon$ by $\pi/2$.

We now seek necessary and sufficient conditions for an effective tensor field $\mathbf{A}^0$ to be the H-limit of a sequence $\mathbf{A}^\varepsilon$ satisfying (H-$\mathscr{S}$).

3. OPTIMAL BOUNDS FOR TWO-DIMENSIONAL COMPOSITES

The mapping

$$F(\lambda_1, \lambda_2) \equiv (\lambda_1 \lambda_2, \lambda_2) \equiv (d, \lambda) \tag{3.1}$$

maps the compact set $\mathscr{S}$ of $(\mathbb{R}_+^*)^2$ onto a compact set $F(\mathscr{S})$ of $(\mathbb{R}_+^*)^2$. Since $\mathscr{S}$ is invariant under interchange of the eigenvalues λ_1 and λ_2, $F(\mathscr{S})$ is invariant under the "*reflection transformation*" T defined for any element (d, λ) of $(\mathbb{R}_+^*)^2$ by

$$T(d, \lambda) \equiv (d, d/\lambda) \tag{3.2}$$

The set $F(\mathscr{S})$ will loosely be referred to as the "(d, λ) representation of Σ", where Σ, the set of oriented conductivity tensors of the components, is defined via (2.1).

The characterization of all possible H-limits of sequences $\mathbf{A}^\varepsilon$ satisfying (H-$\mathscr{S}$) is conveniently addressed in the (d, λ) plane.

3.1. Two-Phase Composites

Consider two oriented materials with conductivity tensors

$$\begin{aligned} \mathbf{A} &= \alpha_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \\ \mathbf{B} &= \beta_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \beta_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \end{aligned} \tag{3.3}$$

and assume that (1.1) is satisfied. The associated set $\mathscr{S} = \mathscr{S}(\Sigma)$ reduces to

$$\mathscr{S}(\mathbf{A}, \mathbf{B}) = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\beta_1, \beta_2), (\beta_2, \beta_1)\} \tag{3.4}$$

and its image by F is simply the set

$$F(\mathscr{S}(\mathbf{A},\mathbf{B}))=\{\mathbf{a}=(\alpha_1\alpha_2,\alpha_2),\ T(\mathbf{a})=(\alpha_1\alpha_2,\alpha_1),\ \mathbf{b}=(\beta_1\beta_2,\beta_2),\ T(\mathbf{b})=(\beta_1\beta_2,\beta_1)\} \tag{3.5}$$

In the three cases considered in the introduction, the region $F(\mathscr{L})$ is invariant under the "reflection transformation" T and is bounded by the vertical line segments $[\mathbf{a}, T(\mathbf{a})]$ and $[\mathbf{b}, T(\mathbf{b})]$ as illustrated in Fig. 3.

In case 1 (Fig. 3a), $F(\mathscr{L})$ is also bounded below by the line segment $[T(\mathbf{a}), T(\mathbf{b})]$ and bounded above by a concave branch of hyperbola joining $\mathbf{a}$ and $\mathbf{b}$, that is, the image under T of the line segment $[T(\mathbf{a}), T(\mathbf{b})]$. In cases 2 and 3 (Figs. 3b and 3c), $F(\mathscr{L})$ is bounded above by the line segment $[\mathbf{a}, \mathbf{b}]$ and is bounded below by a convex branch of hyperbola joining $T(\mathbf{a})$ and $T(\mathbf{b})$, which is the image under T of the line segment $[\mathbf{a}, \mathbf{b}]$. Note that the slope and λ intercept of the line passing through $\mathbf{a}$ and $\mathbf{b}$ have the same sign in case 1, but have opposite signs in cases 2 and 3. Similarly this is true for the line passing through $T(\mathbf{a})$ and $T(\mathbf{b})$.

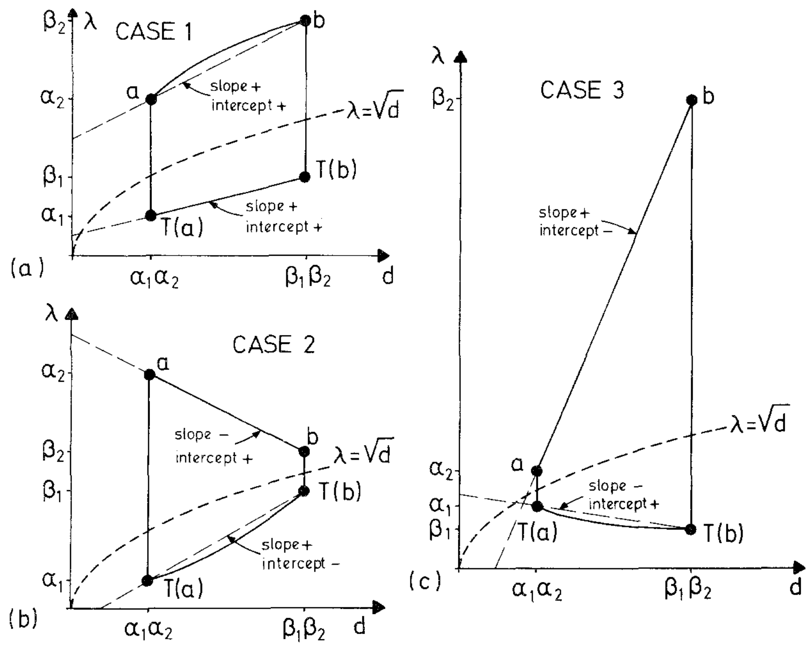

Fig. 3. The region $\mathscr{L}$ in the (λ_1, λ_2) plane maps to a region $F(\mathscr{L})$ with a simple geometrical structure in the $(d, \lambda)=F(\lambda_1, \lambda_2)$ plane. This structure is evident from the examples given here of each of the three cases. In the graphs any point on the curve $\lambda=\sqrt{d}$ is associated with an isotropic material. In each case note the signs of the slope and λ intercept of the line passing through $\mathbf{a}$ and $\mathbf{b}$ and of the line passing through $T(\mathbf{a})$ and $T(\mathbf{b})$.

The following theorem, which provides a complete characterization for the effective tensor fields of all possible composites made of two given anisotropic phases, is the main result of Francfort and Murat.[4]

Theorem 3.1. Let $\mathbf{A}$ and $\mathbf{B}$ be the conductivity tensors defined via (3.3). In each of the cases 1, 2, or 3 the set Λ of all measurable self-adjoint second-order tensor fields $\mathbf{A}^0$ with eigenvalues $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$ such that (λ_1, λ_2) lies in $\mathscr{L}$ almost everywhere is exactly the set of all H-limits of sequences $\mathbf{A}^\varepsilon$ satisfying

$$\mathbf{A}^\varepsilon = {}^t\mathbf{R}^\varepsilon \mathbf{A} \mathbf{R}^\varepsilon \chi^\varepsilon + {}^t\mathbf{R}^\varepsilon \mathbf{B} \mathbf{R}^\varepsilon (1 - \chi^\varepsilon) \tag{3.6}$$

where χ^ε is a measurable sequence of characteristic functions on $\mathbb{R}^2$ and $\mathbf{R}^\varepsilon$ is a measurable sequence of orthogonal matrices on $\mathbb{R}^2$.

Remark 3.1. The set Λ is equivalently defined as the set of all symmetric second-order tensor fields $\mathbf{A}^0$ with eigenvalues (λ_1, λ_2) such that $(\lambda_1 \lambda_2, \lambda_2)$ [and $(\lambda_1 \lambda_2, \lambda_1)$] lie in $F(\mathscr{L})$ almost everywhere.

Remark 3.2. The H-converging sequences $\mathbf{A}^\varepsilon$ that satisfy the conditions of Theorem 3.1 are precisely the sequences $\mathbf{A}^\varepsilon$ that satisfy $(H\text{-}\mathscr{S}\ (\mathbf{A}, \mathbf{B}))$.

Remark 3.3. From now on the sets Λ, $\mathscr{L}$, and $F(\mathscr{L})$ associated with $\mathbf{A}$ and $\mathbf{B}$ are denoted by $\Lambda(\mathbf{A}, \mathbf{B})$, $\mathscr{L}(\mathbf{A}, \mathbf{B})$, and $F(\mathscr{L}(\mathbf{A}, \mathbf{B}))$, respectively.

In view of Remarks 3.1–3.3, the region $F(\mathscr{L}(\mathbf{A}, \mathbf{B}))$ is exactly the "(d, λ) representation" of the possible H-limits of sequences $\mathbf{A}^\varepsilon$ satisfying $(H\text{-}\mathscr{S}(\mathbf{A}, \mathbf{B}))$.

We now investigate composites constructed from arbitrarily many anisotropic materials.

3.2. Multiphase Composites

An arbitrary compact set $\mathscr{S}$ of $(\mathbb{R}^*_+)^2$ symmetric under interchange of the coordinates is considered. The associated set of conductivity tensors defined via (2.1) is denoted by Σ. We denote by $\Lambda(\Sigma)$ the set of all effective conductivity tensor fields $\mathbf{A}^0$ that can be achieved as H-limits of a sequence $\mathbf{A}^\varepsilon$ satisfying $(H\text{-}\mathscr{S})$.

The results of the previous subsection yield the following three remarks, which are obvious from a physical standpoint because for any $\mathbf{A}, \mathbf{B} \in \Sigma$ the set of all composites formed from the phases associated with Σ includes any composite constructed from the phases associated with $\mathbf{A}$ and $\mathbf{B}$ or constructed from any phases that themselves are composites of these two materials.

Remark 3.4. If $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary elements of Σ, then $\Lambda(\mathbf{A}, \mathbf{B})$ is included in $\Lambda(\Sigma)$. Thus we have

$$\bigcup_{(\mathbf{A},\mathbf{B}) \in \Sigma^2} \Lambda(\mathbf{A}, \mathbf{B}) \subset \Lambda(\Sigma). \tag{3.7}$$

A slightly stronger conclusion that pertains to materials that are macroscopically inhomogeneous holds true, namely:

Remarks 3.5. Let $\mathscr{K}_{\mathscr{S}}$ denote the set

$$\mathscr{K}_{\mathscr{S}} \equiv \bigcup_{(\mathbf{A},\mathbf{B}) \in \Sigma^2} \mathscr{L}(\mathbf{A}, \mathbf{B}) \tag{3.8}$$

and let $\mathbf{A}^0$ be a measurable conductivity tensor field such that its eigenvalue pair $(\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))$ lies in $\mathscr{K}_{\mathscr{S}}$ for almost all $\mathbf{x} \in \mathbb{R}^2$. Then $\mathbf{A}^0$ belongs to $\Lambda(\Sigma)$.

A similar result is stated by Tartar (Ref. 9, Section V), for a mixture in fixed volume fraction of two isotropic conducting materials. As explained there, the local character of H-convergence (Ref. 13, Remark 17) permits the analysis to be restricted to a bounded domain Ω. On Ω, $\mathbf{A}^0$ is approximated in the strong topology of $L_1(\Omega)$ by a sequence of functions $\mathbf{A}^n(\mathbf{x})$ whose eigenvalue pairs are constant on *disjoint open* subdomains of Ω and lie in $\mathscr{K}_{\mathscr{S}}$.[3] Since H-convergence results from a metrizable topology and because that topology is weaker than the topology associated with strong convergence in $L_1(\Omega)$, the analysis can be further restricted to conductivity tensor fields with constant eigenvalues lying in $\mathscr{K}_{\mathscr{S}}$. These eigenvalues belong to one of the sets $\mathscr{L}(\mathbf{A}, \mathbf{B})$. By virtue of Theorem 3.1, the corresponding effective conductivity tensor field lies in $\Lambda(\mathbf{A}, \mathbf{B})$, and, in view of Remark 3.4, in $\Lambda(\Sigma)$.

Remark 3.6. Defining

$$\mathscr{F}_{\mathscr{S}} \equiv F(\mathscr{K}_{\mathscr{S}}) = \bigcup_{(\mathbf{A},\mathbf{B}) \in \Sigma^2} F(\mathscr{L}(\mathbf{A}, \mathbf{B})) \tag{3.9}$$

and

$$\Lambda_{\mathscr{S}} \equiv \{\mathbf{A}^0 \mid \mathbf{A}^0(x) \text{ is a measurable symmetric rensor field}$$
$$\text{with } (\lambda_1^0(\mathbf{x})\ \lambda_2^0(\mathbf{x}), \lambda_2^0(\mathbf{x})) \in \mathscr{F}_{\mathscr{S}} \text{ for almost all } \mathbf{x} \in \mathbb{R}^2\} \tag{3.10}$$

[3] The approximation of $\mathbf{A}^0$ by a sequence of functions whose eigenvalues are constant on disjoint open subdomains of Ω is immediate. By projection they can be shown to take their values in the closure of $\mathscr{K}_{\mathscr{S}}$ and by approximation in $\mathscr{K}_{\mathscr{S}}$ itself.

we conclude, with the help of Remark 3.5, that

$$\Lambda_{\mathscr{S}} \subset \Lambda(\Sigma) \tag{3.11}$$

Our ultimate goal is the proof of the inverse inclusion in (3.11). Strictly speaking, we are only in a position to prove the following theorem, which applies to macroscopically homogeneous media. This theorem and its corollary form the central result of the paper.

Theorem 3.2. Let $\mathbf{A}^0$ be a measurable symmetric second-order tensor field on $\mathbb{R}^2$ with constant eigenvalues. If $\mathbf{A}^0$ belongs to $\Lambda(\Sigma)$, then $\mathbf{A}^0$ belongs to $\Lambda_{\mathscr{S}}$.

A geometric characterization, as illustrated in Figs. 2 and 4, of the set

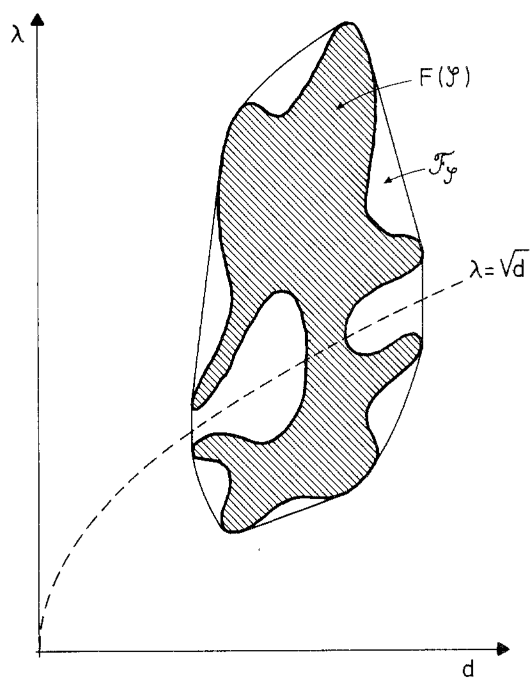

Fig. 4. Construction of the region $\mathscr{F}_{\mathscr{S}}$ for an arbitrarily chosen set of infinitely many materials represented here in the "(d, λ) representation" by the shaded region $F(\mathscr{S})$. The construction is simple: the convex hull $\mathrm{co}(F(\mathscr{S}))$ of $F(\mathscr{S})$ is "reflected" about the curve $\lambda = \sqrt{d}$ via the transformation T to obtain $T(\mathrm{co}(F(\mathscr{S})))$ and the union with $\mathrm{co}(F(\mathscr{S}))$ gives $\mathscr{F}_{\mathscr{S}}$, which is the region bounded by the outermost closed curve in the figure. This set is the "(d, λ) representation" of the set of all possible effective tensors of two-dimensional composites formed from the given phases. See also Fig. 2, which depicts the construction of $\mathscr{F}_{\mathscr{S}}$ for three-phase composites.

of all such effective tensors $\mathbf{A}^0$ is also derived in the proof of Theorem 3.2. Specifically, we obtain the following result:

Corollary 3.1. Let $\mathbf{A}^0$ be as in Theorem 3.2 and let λ_1^0 and λ_2^0 denote its eigenvalues. Then $\mathbf{A}^0$ belongs to $\Lambda(\Sigma)$ if and only if $(\lambda_1^0\lambda_2^0, \lambda_1^0)$ and $(\lambda_1^0\lambda_2^0, \lambda_2^0)$ lie in $\mathrm{co}(F(\mathscr{S})) \cup T(\mathrm{co}(F(\mathscr{S})))$, where $F(\mathscr{S})$ is the "(d, λ) representation of Σ," $\mathrm{co}(F(\mathscr{S}))$ denotes the closed convex hull of $F(\mathscr{S})$, and T is the "reflection transformation" defined by (3.2).

Remark 3.7. In fact it is true that

$$\Lambda(\Sigma) = \Lambda_{\mathscr{S}} \tag{3.12}$$

but a complete proof requires an extension of the result of Theorem 3.2 to any measurable symmetric second-order tensor field $\mathbf{A}^0$ lying in $\Lambda(\Sigma)$. The extension may be accomplished with the help of the following result, communicated by R. V. Kohn (cf. Kohn and Dal Maso[14]).

A subset of S of $L_1(\mathbb{R}^m, \mathbb{R}^p)$, where m and p are arbitrary integers, is such that all its elements lie almost everywhere in a closed subset S' of $\mathbb{R}^p$ if and only if it is translation invariant, closed in $L_1(\mathbb{R}^n, \mathbb{R}^p)$, and decomposable.

For $\Lambda(\Sigma)$ these properties are easily verified, at least locally, and we may conclude to the existence of a closed subset Σ' of all symmetric matrices such that

$$\Lambda(\Sigma) = L_\infty(\mathbb{R}^2, \Sigma') \tag{3.13}$$

Theorem 3.2 then shows that Σ' is included in $\Lambda_{\mathscr{S}}$ and, by the definition of $\Lambda_{\mathscr{S}}$, that $L_\infty(\mathbb{R}^2, \Sigma')$ is also included in $\Lambda_{\mathscr{S}}$. This was the result we sought.

Remark 3.8. The case of periodic homogenization (see, for example, Bensoussan *et al.*[15]), which yields homogeneous H-limits, is covered by Theorem 3.2.

Remark 3.9. Remark 3.6, Theorem 3.2, and Corollary 3.1 imply that $\mathscr{F}_{\mathscr{S}}$ is closed and

$$\mathscr{F}_{\mathscr{S}} = \mathrm{co}(F(\mathscr{S})) \cup T(\mathrm{co}(F(\mathscr{S}))) \tag{3.14}$$

Remark 3.10. A result of the type obtained in Theorem 3.2 and even of the type (3.12) is claimed without proof in Ref. 3, Section 4.

Proof of Theorem 3.2 and Corollary 3.1. The proof is established in two steps, of which the second is the most substantial.

First let us prove

$$\mathrm{co}(F(\mathscr{S})) \cup T(\mathrm{co}(F(\mathscr{S}))) \subset \mathscr{F}_{\mathscr{S}} \tag{3.15}$$

To this end consider an arbitrary point $\mathbf{p}$ of $\mathrm{co}(F(\mathcal{S}))$. It lies in the convex hull of a finite number N of points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ of $F(\mathcal{S})$. (It suffices, of course, to take $N=3$). Possible addition of the images of those points by the "reflection transformation" T enables us to assume that $\mathbf{p}$ lies in the convex hull of N pairs of points $\mathbf{a}_i$ and $T(\mathbf{a}_i)$ of $F(\mathcal{S})$. That hull is the union over all pairs $\mathbf{a}_i$ and $\mathbf{a}_j$ of the convex trapezoids with vertices $(\mathbf{a}_i, \mathbf{a}_j, T(\mathbf{a}_i), T(\mathbf{a}_j))$. Now, the properties of the region $F(\mathcal{L})$ described in Section 3.1 imply that the convex trapezoid with vertices $(\mathbf{a}, \mathbf{b}, T(\mathbf{a}), T(\mathbf{b}))$ is included in $F(\mathcal{L}(\mathbf{A}, \mathbf{B}))$ for any conductivity tensors $\mathbf{A}$ and $\mathbf{B}$, represented by points $\mathbf{a}$ and $T(\mathbf{a})$, and $\mathbf{b}$ and $T(\mathbf{b})$ in the "(d, λ) representation." Hence, each of the trapezoids with vertices $(\mathbf{a}_i, \mathbf{a}_j, T(\mathbf{a}_i), T(\mathbf{a}_j))$ is included in $\mathcal{F}_{\mathcal{S}}$. Since $\mathcal{F}_{\mathcal{S}}$ is invariant under the"reflection transformation" T, $T(\mathbf{p})$ also belongs to $\mathcal{F}_{\mathcal{S}}$. The proof of (3.15) is complete.

Let $\mathbf{A}^0$ be a conductivity tensor field of the form described in the statement of Theorem 2.2 and let $\mathbf{p}^0$ denote its "(d, λ) representation." If λ_1^0, λ_2^0 denote the eigenvalues of $\mathbf{A}^0$, we may with no loss of generality assume

$$\lambda_1^0 \leqslant \lambda_2^0 \tag{3.16}$$

and that $\mathbf{p}^0$ corresponds to the point with λ-coordinate λ_2^0.

In the second step of the proof we suppose that both $\mathbf{p}^0$ and $T(\mathbf{p}^0)$ do not lie in $\mathrm{co}(F(\mathcal{S}))$ and then reach a contradiction.

Since $\mathcal{S}$ is compact, $\mathrm{co}(F(\mathcal{S}))$ is compact, and since $\mathbf{p}^0$ and $T(\mathbf{p}^0)$ do not belong to $\mathrm{co}(F(\mathcal{S}))$, we deduce the existence of two nonvertical lines l and l' that respectively separate $\mathbf{p}^0$ and $T(\mathbf{p}^0)$ from $\mathrm{co}(F(\mathcal{S}))$, with $\mathbf{p}^0$ above l and $T(\mathbf{p}^0)$ below l'. [When $\mathbf{p}^0$ lies above $\mathrm{co}(F(\mathcal{S}))$ these lines are necessarily nonvertical, with $\mathbf{p}^0$ above l and $T(\mathbf{p}^0)$ below l'. Otherwise there is a vertical line separating $\mathbf{p}^0$ from $\mathrm{co}(F(\mathcal{S}))$, which may be slightly rotated in either direction, without intersecting the compact set $\mathrm{co}(F(\mathcal{S}))$, to obtain l and l' with $\mathbf{p}^0$ above l and $T(\mathbf{p}^0)$ below l'.] Let us define

$$d^- \equiv \min_{(\lambda_1 \lambda_2) \in \mathcal{S}} \lambda_1 \lambda_2, \qquad d^+ \equiv \max_{(\lambda_1 \lambda_2) \in \mathcal{S}} \lambda_1 \lambda_2 \tag{3.17}$$

and let us denote the points of intersection of l with the vertical lines $d = d^-$ and $d = d^+$ by $\mathbf{a}_l$ and $\mathbf{b}_l$, respectively (see Fig. 5a). These points, like l, lie above $\mathrm{co}(F(\mathcal{S}))$.

Now consider the set $F(\mathcal{L}(\mathbf{A}_l, \mathbf{B}_l))$, where $\mathbf{A}_l$ and $\mathbf{B}_l$ are the conductivity tensors of two fictitious materials whose "(d, λ) representations" are $\mathbf{a}_l$ and $T(\mathbf{a}_l)$, and $\mathbf{b}_l$ and $T(\mathbf{b}_l)$. The set $F(\mathcal{S})$ lies below the line segment $[\mathbf{a}_l, \mathbf{b}_l]$ and hence $T(F(\mathcal{S})) = F(\mathcal{S})$ lies above $T([\mathbf{a}_l, \mathbf{b}_l])$. But the region bounded by the line segments $[\mathbf{a}_l, \mathbf{b}_l]$, $[\mathbf{a}_l, T(\mathbf{a}_1)]$, and $[\mathbf{b}_l, T(\mathbf{b}_l)]$ and by the arc of hyperbola $T([\mathbf{a}_l, \mathbf{b}_l])$ is included in $F(\mathcal{L}(\mathbf{A}_l, \mathbf{B}_l))$. Thus, $F(\mathcal{S})$

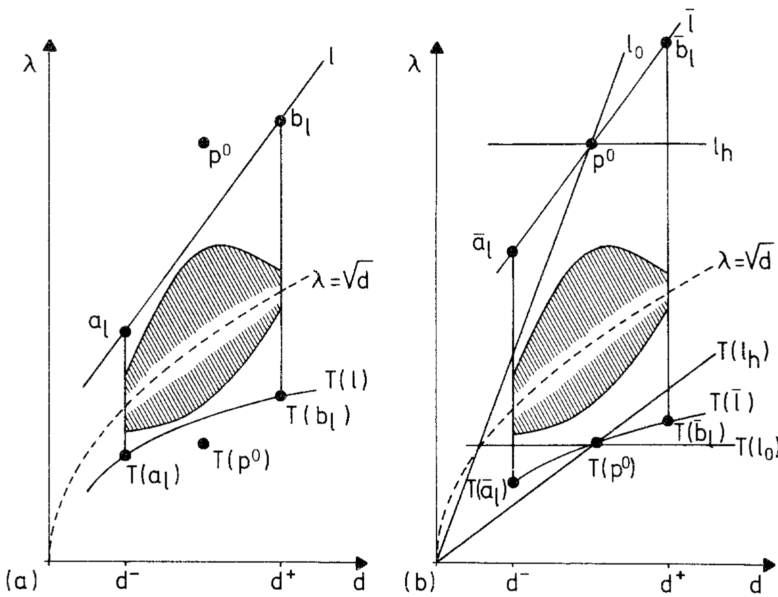

Fig. 5. Sketches showing the various lines l, $\bar{l}$, l_0, and l_h and the various points $\mathbf{p}^0$, $\mathbf{a}_l$, $\mathbf{b}_l$, $\bar{\mathbf{a}}_l$, $\bar{\mathbf{b}}_l$, and their images under the "reflection mapping" T. These are introduced in the proof of Theorem 3.2 and Corollary 3.1. The shaded regions represent $\mathrm{co}(F(\mathscr{S}))$ for an arbitrary set $\mathscr{S}$.

lies inside $F(\mathscr{L}(\mathbf{A}_l, \mathbf{B}_l))$ and, since H-convergence derives from a metrizable topology, $\Lambda(\Sigma)$ is included in $\Lambda(\mathbf{A}_l, \mathbf{B}_l)$. In more physical terms, each individual phase of the original set can be regarded as a composite resulting from mixing the materials with conductivity tensors $\mathbf{A}_l$ and $\mathbf{B}_l$. Thus, any composite formed from phases in the original set may also be viewed as a composite resulting from mixing the materials with conductivity tensors $\mathbf{A}_l$ and $\mathbf{B}_l$. We thereby conclude that

$$\mathbf{p}^0 \in F(\mathscr{L}(\mathbf{A}_l, \mathbf{B}_l)) \tag{3.18}$$

In cases 2 and 3, discussed in Section 3.1 and illustrated in Fig. 3, $F(\mathscr{L}(\mathbf{A}_l, \mathbf{B}_l))$ is bounded above by the line segment $[\mathbf{a}_l, \mathbf{b}_l]$ and hence by l; $\mathbf{p}^0$ cannot belong to $F(\mathscr{L}(\mathbf{A}_l, \mathbf{B}_l))$ and we have reached a contradiction with (3.18). In the remaining possibility, case 1, l always has nonnegative slope and nonnegative λ intercept. Hence the slope of l lies between the slope of the horizontal line l_h through $\mathbf{p}^0$ and the slope of the line l_0 joining $\mathbf{p}^0$ to the origin. Let $\mathscr{C}$ denote the closed double cone of vertex $\mathbf{p}^0$ bounded by the lines l_0 and l_h. By translating l parallel to itself, we obtain a line $\bar{l}$, sketched in Fig. 5b, that satisfies

$$\mathbf{p}^0 \in \bar{l}, \qquad \bar{l} \cap \operatorname{co}(F(\mathscr{S})) = \varnothing, \qquad \bar{l} \subset \mathscr{C} \tag{3.19}$$

Note that $T(l_h)$ and $T(l_0)$ are lines and not hyperbolas as they would be for the image under the reflection transformation T of an arbitrary line. Hence, the image $T(\mathscr{C})$ of $\mathscr{C}$ under T is also a closed double cone of vertex $T(\mathbf{p}^0)$ bounded by the line $T(l_h)$ joining $T(\mathbf{p}^0)$ to the origin and by the horizontal line $T(l_0)$.

Let us denote by $\bar{\mathbf{a}}_l$ and $\bar{\mathbf{b}}_l$ the intersections of $\bar{l}$ with the vertical lines $d = d^-$ and $d = d^+$, respectively. The set $F(\mathscr{S})$ lies strictly below the line segment $[\bar{\mathbf{a}}_l, \bar{\mathbf{b}}_l]$ and hence it also lies strictly above the curve $T([\bar{\mathbf{a}}_l, \mathbf{b}_l])$. Since $\bar{l}$ lies inside $\mathscr{C}$, $T([\bar{\mathbf{a}}_l, \bar{\mathbf{b}}_l])$ is easily checked to be a concave branch of hyperbola lying inside $T(\mathscr{C})$ and passing through $T(\mathbf{p}^0)$ as shown in Fig. 5b.

We now take l' and translate it parallel to itself to obtain a line $\bar{l}'$ that passes through $T(\mathbf{p}^0)$ and does not intersect $\operatorname{co}(F(\mathscr{S}))$. If $\bar{l}'$ lies outside the cone $T(\mathscr{C})$, the concavity of $T([\bar{\mathbf{a}}_l, \bar{\mathbf{b}}_l])$ implies that $F(\mathscr{S})$ lies strictly above either the line through $T(\bar{\mathbf{a}}_l)$ and $T(\mathbf{p}^0)$ or the line through $T(\bar{\mathbf{b}}_l)$ and $T(\mathbf{p}^0)$ (see Fig. 6). That line below $F(\mathscr{S})$ is denoted by l^*. In either case, l^* lies within the cone $T(\mathscr{C})$ and lies below $F(\mathscr{S})$, and hence does not intersect

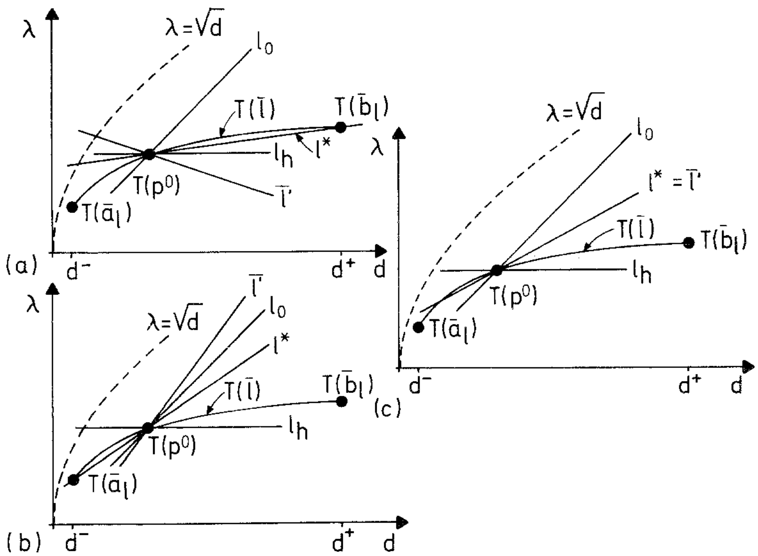

Fig. 6. The construction of the line l^* depends on where the line $\bar{l}'$ is located in relation to the cone $T(\mathscr{C})$ bounded by the lines $T(l_0)$ and $T(l_h)$. The three examples depicted here typify all possibilities.

$\mathrm{co}(F(\mathscr{S}))$. If $\bar{l}'$ lies inside $T(\mathscr{C})$, we identify l^* with $\bar{l}'$. All possibilities result in a line l^* satisfying

$$T(\mathbf{p}^0) \in l^*, \qquad l^* \cap \mathrm{co}(F(\mathscr{S})) = \varnothing, \qquad l^* \subset T(\mathscr{C}) \tag{3.20}$$

Since $F(\mathscr{S})$ is compact, we can translate l^* by a small amount parallel to itself but without intersecting $\mathrm{co}(F(\mathscr{S}))$, to obtain a line $\bar{l}^*$ with nonnegative slope and nonnegative λ intercept that separates $T(\mathbf{p}^0)$ from $F(\mathscr{S})$.

But an argument similar to the one used to prove l has nonnegative slope and nonnegative λ intercept would show that any line such as $\bar{l}^*$ separating $T(\mathbf{p}^0)$ from $F(\mathscr{S})$ cannot have both nonnegative slope and nonnegative λ intercept, in contradiction with the above result.

Thus, $\mathbf{p}^0$ or $T(\mathbf{p}^0)$ belongs to $\mathrm{co}(F(\mathscr{S}))$, which proves the "only if" part of Corollary 3.1. Recalling (3.15) then completes the proof of Theorem 3.2 and the "if" part of Corollary 3.1. ▌

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